Algebraic Methods	January 18, 2017
Exercise 4	
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Instructions: Please do not copy anyone else's solution. You are allowed to consult with other classmates if you thought about the problem yourself first, but you should only ask for the idea of the solution rather than copying the entire solution. And, most importantly, you should give credit to the classmate with whom you discussed the solution.

1. Let $f_1, ..., f_m \in \mathbb{F}[x_1, ..., x_n]$ be polynomials over the field $\mathbb{F}$. For a point $u \in \mathbb{F}^n$ we assign a vector $\chi(u) \in \{0,1\}^m$ in the following way: $\chi(u)_i = 1$ iff $f_i(u) = 0$. Let $S \subseteq \{0,1\}^m$ be the set of all such vectors.

   (a) Show that if the f_i's are linear functions then the number of *different* vectors in S is at most
   $$\sum_{i=0}^{n} \binom{m}{i}.$$

   (b) Let $\deg(f_i) = d_i$. Prove that the number of *different* vectors in S is at most
   $$\binom{n + \sum_{i=1}^{m} d_i}{n}.$$

   Hint: For every vector $v_i \in S$ construct a polynomial g_i and a vector x_i such that $g_i(x_i) \neq 0$ and $\forall j < i : g_i(x_j) = 0$ (choose the order of S carefully). Show that the g_i are linearly independent and belong to a low dimensional space.

2. Here we shall slightly generalize a lemma that we saw when proving the combinatorial Nullstellensatz result. Let $M(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{e_i}$ be a monomial. Let $f(x) \in \mathbb{F}[x_1, \ldots, x_n]$ be a polynomial. We say that M is not dominated in f if M is one of the monomials in f and in any other monomial $M' = \prod_{i=1}^{n} x_i^{d_i}$ appearing in f we must have $d_i < e_i$ for some i. Notice that even if M is not dominated in f then we may still have that the degree of f is much larger than the degree of M and also that the degree of each individual variable in f is larger than its degree in M.

   Let $S_1, \ldots, S_n \subseteq \mathbb{F}$ be sets of size $|S_i| > e_i$. Prove that if the monomial $M(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i^{e_i}$ is not dominated in f then there is some point $\alpha \in S_1 \times S_2 \times \cdots \times S_n$ such that $f(\alpha) \neq 0$.

3. **Not for submission:** Fourier representation as a multilinear polynomial.

   Let $T : \{0,1\} \to \{-1,1\}$ be the following transformation: $T(0) = 1$ and $T(1) = -1$. Let $f : \{0,1\}^n \to \{0,1\}$ and $f' : \{-1,1\}^n \to \{-1,1\}$ be two functions such that for every $x \in \{-1,1\}^n$
   $$f'(x) = T\left(f\left(T^{-1}(x_1), ..., T^{-1}(x_n)\right)\right).$$

(a) Show that T is a linear function.

(b) Show that $\{\chi_S\}_{S\subseteq[n]}$, where $\chi_S = \prod_{i\in S} x_i$, is an orthonormal basis for functions $\{g : \{-1,1\}^n \to \{-1,1\}\}$.

(c) Let $F(x) = \sum_{S\subseteq[n]} a_s \prod_{i\in S} x_i$ be a multilinear polynomial over the reals such that for any $x \in \{-1,1\}^n$, $F(x) = f'(x)$. Prove that $\widehat{(1-2f)}(1_S) = a_s$.

(d) Prove that the Fourier degree of f and f' (according to the different Fourier basis) is the same.

(e) Prove that $\deg_{\mathbb{F}_2}(f) \leq \deg_{\mathbb{R}}(F)$. In words, the degree of the polynomial representing f over $\mathbb{F}_2$ is never larger than the degree of the multilinear polynomial representing f over $\mathbb{R}$.

4. Let $S \subseteq \{0,1\}^n$. An S threshold function $f : \{0,1\}^n \to \{-1,1\}$ is a function of the form $f(x) = \text{sign}(F(x))$, where $F : \{0,1\}^n \to \mathbb{R} \setminus \{0\}$ is such that for any $\alpha \notin S$, it holds that $\hat{F}(\alpha) = 0$. The following lemma links the Fourier coefficients of f to those of F.

Lemma 1. *Let $f : \{0,1\}^n \to \{-1,1\}$ and $F : \{0,1\}^n \to \mathbb{R} \setminus \{0\}$ be two functions. Then,*

$$\forall x \in \{0,1\}^n \quad f(x) = sign(F(x))$$

if and only if

$$\sum_{x\in\{0,1\}^n} |F(x)| = 2^n \sum_{\alpha} \hat{f}(\alpha)\hat{F}(\alpha)$$

(a) Prove Lemma 1.

(b) Let $S \subseteq \{0,1\}^n$ be such that $(1,1,...,1) \notin S$. Show that the parity function is not an S threshold function.

(c) Fix $S \subseteq \{0,1\}^n$. Let $f : \{0,1\}^n \to \{-1,1\}$ be an S threshold function and $g : \{0,1\}^n \to \{-1,1\}$ be another function (not necessarily a threshold function). Prove that $f = g$ iff for every $\alpha \in S$, $\hat{f}(\alpha) = \hat{g}(\alpha)$.

(d) Let $f, g : \{0,1\}^n \to \{-1,1\}$ be two functions. Let $S \subseteq \{0,1\}^n$ be a set such that $\forall \alpha \in S$ it holds that $\hat{f}(\alpha) = \hat{g}(\alpha)$. Show that either both f and g are S threshold functions or both are not S threshold functions.

5. Sensitivity.

For $x \in \{0,1\}^n$ denote with x^i the vector that we get after flipping the i'th coordinate of x. For $f : \{0,1\}^n \to \{-1,1\}$ and $x \in \{0,1\}^n$ define the sensitivity of f at x to be

$$\mathrm{s}_f(x) = \left|\{\, i \mid f(x) \neq f(x^i)\}\right|.$$

Let the average sensitivity of f be

$$\mathrm{as}(f) = \frac{1}{2^n} \sum_{x\in\{0,1\}^n} \mathrm{s}_f(x).$$

(a) Show that $\mathrm{as}(f) = \sum_{S \subset [n]} |S| \cdot \hat{f}(1_S)^2$.

(b) Prove that if $\mathrm{as}(f) = 1$ and f is balanced (that is, $\Pr_{x \in_u \{0,1\}^n}[f(x) = 1] = \frac{1}{2}$) then f is a dictatorship, i.e., f depends on only one variable.

6. Influence.

For $f : \{0,1\}^n \to \{-1,1\}$ we define the influence of the i'th variable as

$$\mathrm{Inf}_i(f) = \Pr[f(x) \neq f(x^i)]$$

where the probability is over a uniform choice of $x \in \{0,1\}^n$. As before, let $\deg_{\mathbb{R}}(f)$ be the degree of the multilinear polynomial representing f over $\mathbb{R}$.

(a) Prove that

$$\sum_{i=1}^{n} \mathrm{Inf}_i(f) = \mathrm{as}(f).$$

(b) Show that

$$\sum_{i=1}^{n} \mathrm{Inf}_i(f) \leq \deg_{\mathbb{R}}(f).$$

(c) Prove that if f depends on the i'th variable then

$$\frac{1}{2^{\deg_{\mathbb{R}}(f)}} \leq \mathrm{Inf}_i(f).$$

(d) Conclude that if f depends on all n variables then $\deg_{\mathbb{R}}(f) \geq \log n - O(\log \log n)$.

7. No small Fourier coefficients means (relatively) low degree over $\mathbb{F}_2$.

Let $f : \{0,1\}^n \to \{-1,1\}$ be a non-constant function satisfying that for all $\alpha \in \{0,1\}^n$ with $1 \leq \mathrm{wt}(\alpha) < t$, $\hat{f}(\alpha) = 0$.

(a) Prove that if $\hat{f}(\overline{0}) \neq 0$ then $t \leq \frac{2}{3}n$, and that this is tight.
Hint: Consider f^2.

(b) Assume that $\hat{f}(\overline{0}) = 0$ and that f is not the parity function nor its negation. Prove that f has degree at most $n - t$ as an $\mathbb{F}_2$ polynomial (when we think of it as a function to $\{0,1\}$ via the mapping $-1 \to 1$, $1 \to 0$). Show that this result is tight.
Hint: Consider xoring f with the parity function.