Algebraic Methods	November 18, 2016
Exercise 1	
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Instructions: Please do not copy anyone else's solution. You are allowed to consult with other classmates if you thought about the problem yourself first, but you should only ask for the idea of the solution rather than copying the entire solution. And, most importantly, you should give credit to the classmate with whom you discussed the solution.

1. Show that if $A_1, \ldots, A_m \subseteq [n]$ are such that

   (a) for every i, $|A_i|$ is even, and

   (b) for every $i \neq j$, $|A_i \cap A_j|$ is odd

   then $m \leq n$. Further, if n is even then $m < n$.

2. Prove that if $v_1, \ldots, v_m \in \mathbb{R}^n$ and $\delta \in \mathbb{R}$ are such that for any $i \neq j$ $\|v_i - v_j\|_2 = \delta$ then $m \leq n+1$. Is this bound tight?

3. Let p be a prime number and $L_1, \ldots, L_m$ sets of integer such that for every i, $|L_i| = s$. Assume $\mathcal{F} = \{A_1, \ldots, A_m\}$ is a family of subsets of $[n]$ such that

   - $|A_i| \notin L_i \mod p$ for $i = 1, \ldots m$.
   - For $j < i$ $|A_i \cap A_j| \in L_i \mod p$

   Prove that
   $$m \leq \binom{n}{s} + \binom{n}{s-1} + \ldots + \binom{n}{0}.$$

4. Let G be a graph and A_G its adjacency matrix. Denote with $\mathrm{bc}(G)$ the minimal number of disjoint bipartite cliques that cover all edges of G (recall that in a disjoint cover, each edge of G is covered by exactly one bipartite clique). In class we proved that $\mathrm{bc}(K_n) = n-1$.

   (a) Assume that A_G has p positive eigenvalues and q negative eigenvalues. Prove that $\mathrm{bc}(G) \geq \max(p, q)$.

   (b) Show that this implies the Graham-Pollak theorem.

5. Let G be a d-regular graph on n vertices. Let A_G be its adjacency matrix: $(a_{i,j}) = 1$ iff (i, j) is an edge in G.

   (a) Show that d is an eigenvalue of A_G. Give an eigenvector with eigenvalue d.

   (b) Show that the multiplicity of d is 1 iff G is connected.

   (c) Show that $-d$ is an eigenvalue iff G has a bipartite connected component.

6. Let $A = (a_{i,j})$ be an $n \times n$ real, symmetric matrix. Prove that

$$\operatorname{rank}(A) \geq \frac{\operatorname{Tr}(A)^2}{\operatorname{Tr}(A^2)},$$

where $\operatorname{Tr}(A)$ is the trace of A.

7. Let $B = (b_{i,j})$ by an $n \times n$ real matrix.

   (a) Prove that if for every i, $|b_{i,i}| > \sum_{j \neq i} |b_{i,j}|$, then B is of full rank. Conclude that if $b_{i,i} = 1$ for all i and $|b_{i,j}| \leq \frac{1}{n}$ for all distinct indices i, j, then the rank of B is n.

   Suppose we relax the conditions above, and only assume that each diagonal entry is, in absolute value, at least $\frac{1}{2}$ and the absolute value of each other entry is at most ϵ. In this case one can also establish a lower bound for the rank of B, as stated in the following theorem, which you'll prove in this exercise.

   Theorem 1. *There exists an absolute positive constant c so that the following holds. Let B be an $n \times n$ real matrix with $|b_{i,i}| \geq \frac{1}{2}$ for all i and $|b_{i,j}| \leq \epsilon$ for all $i \neq j$, where $\frac{1}{2\sqrt{n}} \leq \epsilon < \frac{1}{4}$. Then the rank of B satisfies*

$$\operatorname{rank}(B) \geq \frac{c}{\epsilon^2 \log \frac{1}{\epsilon}} \log n.$$

   To prove the theorem we'll prove the following lemma:

   Lemma 2. *Let $B = (b_{i,j})$ be an $n \times n$ matrix of rank d, and let $P(x)$ be an arbitrary univariate polynomial of degree k. Define $P(B)$ to be the matrix $P(B)_{i,j} = P(b_{i,j})$ (i.e., we apply P to each entry of B). Then, the rank of the $n \times n$ matrix $P(B)$ is at most $\binom{k+d}{k}$. Moreover, if $P(x) = x^k$ then the rank of $P(B)$ is at most $\binom{k+d-1}{k}$.*

   (b) Let $\mathbf{v_1} = (v_{1,j})_{j=1}^n, \ldots, \mathbf{v_d} = (v_{d,j})_{j=1}^n$ be a basis for the row-space of B. Prove that the set of vectors $(v_{1,j}^{k_1} \cdot v_{2,j}^{k_2} \cdots v_{d,j}^{k_d})_{j=1}^n$ where $(k_1, \ldots, k_d)$ range over all non-negative integers whose sum is at most k, span the rows of the matrix $P(B)$.

   (c) Prove lemma 2.

   It is more convenient to first prove the following variant of theorem 1.

   Theorem 3. *There exists an absolute positive constant c so that the following holds. Let B be an $n \times n$ real matrix with $b_{i,i} = 1$ for all i and $|b_{i,j}| \leq \epsilon$ for all $i \neq j$, where $\frac{1}{\sqrt{n}} \leq \epsilon < \frac{1}{2}$. If the rank of B is d, then*

$$d \geq \frac{c}{\epsilon^2 \log \frac{1}{\epsilon}} \log n$$

   (d) Looking at $(B + B^t)/2$, explain why can we assume that B is symmetric for the purpose of Theorem 3.

(e) If $\epsilon \leq 1/n^{\delta}$ for some fixed $\delta > 0$, apply Question (6) on the top-left submatrix of B of size $\lfloor \frac{1}{\epsilon^2} \rfloor$ by $\lfloor \frac{1}{\epsilon^2} \rfloor$ to prove the theorem.

We are left with the (more interesting) case where $\epsilon > 1/n^{\delta}$ for some fixed, small $\delta > 0$. Put $k = \lfloor \frac{\log n}{2 \log 1/\epsilon} \rfloor$, $n' = \lfloor \frac{1}{\epsilon^{2k}} \rfloor$. Note that $n' \leq n$.

Consider the matrix $B' = (b_{i,j}^k)_{i,j \leq n'}$. The idea behind taking the entries of the matrix to the power of k is that the small entries, located outside of the main diagonal, will become small, whereas the diagonal entries will remain the same.

(f) Prove that $\frac{n'}{2} \leq \operatorname{rank}(B') \leq \binom{d+k}{k}$

(g) Prove Theorem 3.

We can now conclude Theorem 1 from Theorem 3.

(h) Prove Theorem 1
Hint: Consider multiplying by the diagonal matrix C, where $c_{i,i} = 1/b_{i,i}$.