Foundation of Cryptography, Lecture 3
Hardcore Predicates for Any One-way Function

Handout Mode

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Informal discussion

f is one-way $\implies$ predicting x from $f(x)$ is hard.

But predicting parts of x might be easy.

e.g., let f be a OWF then $g(x, w) = (f(x), w)$ is one-way

Can we find a function of x that is totally unpredictable — looks uniform — given $f(x)$?

Such functions have many cryptographic applications

Formal definition

Definition 1 (hardcore predicates)

A poly-time computable $b\colon \{0,1\}^n \mapsto \{0,1\}$ is an hardcore predicate of $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, if

$$\Pr_{x \leftarrow \{0,1\}^n}[\mathsf{P}(f(x)) = b(x)] \leq \frac{1}{2} + \operatorname{neg}(n)$$

for any PPT P.

- Does any OWF has such a predicate?
- Is there a generic hardcore predicate for all one-way functions?

  Let f be a OWF and let b be a predicate, then $g(x) = (f(x), b(x))$ is one-way.
- Does the existence of hardcore predicate for f implies that f is one-way?

  Consider $f(x,y) = x$, then $b(x,y) = y$ is a hardcore predicate for f

  Answer to above is positive, in case f is one-to-one

Weak hardcore predicates

For $x \in \{0,1\}^n$ and $i \in [n]$, let x_i be the i'th bit of x.

Theorem 2

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, define $g\colon \{0,1\}^n \times [n] \mapsto \{0,1\}^n \times [n]$ by

$$g(x,i) = f(x), i$$

Assuming f is one way, then

$$\Pr_{x \leftarrow \{0,1\}^n, i \leftarrow [n]}[\mathsf{A}(f(x), i) = x_i] \leq 1 - 1/2n$$

for any PPT A*.*

Proof: ?

We can now construct an hardcore predicate “for" f:

1. Construct a weak hardcore predicate for g (i.e., $b(x,i) := x_i$).
2. Amplify it into a (strong) hardcore predicate for g^t via parallel repetition

The resulting predicate is not for f but for (the one-way function) g^t ...

The Goldreich-Levin Hardcore predicate

For $x, r \in \{0,1\}^n$, let $\langle x, r \rangle_2 := (\sum_{i=1}^{n} x_i \cdot r_i) \bmod 2 = \bigoplus_{i=1}^{n} x_i \cdot r_i$.

Theorem 3 (Goldreich-Levin)

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, define $g\colon \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}^n \times \{0,1\}^n$ as $g(x,r) = (f(x), r)$.

If f is one-way, then $b(x,r) := \langle x, r \rangle_2$ is an hardcore predicate of g.

- Note that if f is one-to-one, then so is g.
- A slight cheat, b is defined for g and not for the original OWF f

Proof by reduction: a PPT A for predicting $b(x,r)$ “too well" from $(f(x), r)$, implies an inverter for f

Section 1

Proving GL – The information theoretic case

Min entropy

Definition 4 (min-entropy)

The min entropy of a random variable (or distribution) X, is defined as

$$\mathrm{H}_\infty(X) := \min_{y \in \mathrm{Supp}(X)} \log \frac{1}{\Pr_X[y]}.$$

Examples:

- Z is uniform over a set of size 2^k.
- $Z = X \mid_{f(X)=y}$, where $f \colon \{0,1\}^n \mapsto \{0,1\}^n$ is 2^k to 1 , $y \in f(\{0,1\}^n) := \{f(x) \colon x \in \{0,1\}^n\}$ and X is uniform over $\{0,1\}^n$.

  Equivalently, $X \leftarrow f^{-1}(y)$.

In both cases, $\mathrm{H}_\infty(Z) = k$.

Pairwise independent hashing

Definition 5 (pairwise independent function family)

A function family $\mathcal{H} = \{h \colon \{0,1\}^n \mapsto \{0,1\}^m\}$ is pairwise independent, if $\forall$ $x \neq x' \in \{0,1\}^n$ and $y, y' \in \{0,1\}^m$, it holds that $\Pr_{h \leftarrow \mathcal{H}}[h(x) = y \wedge h(x') = y')] = 2^{-2m}$.

Lemma 6 (leftover hash lemma)

Let X be a rv over $\{0,1\}^n$ with $\mathrm{H}_\infty(X) \geq k$ and let $\mathcal{H} = \{h \colon \{0,1\}^n \mapsto \{0,1\}^m\}$ be pairwise independent, then

$$\mathrm{SD}((H, H(X)), (H, U_m)) \leq 2^{(m-k-2))/2},$$

where H is uniformly distributed over $\mathcal{H}$ and U_m is uniformly distributed over $\{0,1\}^m$.

See proof here, page 13.

Efficient function families

Definition 7 (efficient function families)

An ensemble of function families $\mathcal{F} = \{\mathcal{F}_n\}_{n \in \mathbb{N}}$ is efficient, if

Samplable. Exists PPT that given 1^n, outputs (the description of) a uniform element in $\mathcal{F}_n$.

Efficient. Exists poly-time algorithm that given $x \in \{0,1\}^n$ and (a description of) $f \in \mathcal{F}_n$, outputs $f(x)$.

Proving GL for compressing functions

Definition 8

Function $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ is $d(n)$ regular, if $\left|f^{-1}(y)\right| = d(n)$ for every $y \in f(\{0,1\}^n)$.

Lemma 9

Let $f\colon \{0,1\}^n \mapsto \{0,1\}^n$ be a $d(n) \in 2^{\omega(\log n)}$ regular function, and let $\mathcal{H} = \{\mathcal{H}_n\}$ be an efficient family of Boolean pairwise independent functions over $\{0,1\}^n$. Define $g\colon \{0,1\}^n \times \mathcal{H}_n \mapsto \{0,1\}^n \times \mathcal{H}_n$ as

$$g(x,h) = (f(x),h),$$

then $b(x,h) = h(x)$ is an hardcore predicate of g.

How does it relate to Goldreich-Levin?
$\{\mathcal{H}_n = \{b_r(\cdot) = b(r,\cdot)\}_{r\in\{0,1\}^n}\}$ is (almost) pairwise independent.

Proving Lemma 9

The lemma follows by the next claim (?)

Claim 10

$\mathrm{SD}\left((f(U_n), H, H(U_n)), (f(U_n), H, U_1)\right) = \mathrm{neg}(n)$, where $H = H_n$ is uniformly distributed over $\mathcal{H}_n$.

Proving the claim. For $y \in f(\{0,1\}^n)$, let X_y be uniformly distributed over $f^{-1}(y) := \{x \in \{0,1\}^n \colon f(x) = y\}$. Compute

$$\begin{aligned}
&\mathrm{SD}((f(U_n), H, H(U_n)), (f(U_n), H, U_1)) \\
&= \mathop{\mathrm{E}}_{y \leftarrow f(U_n)} \left[\mathrm{SD}((f(U_n), H, H(U_n)|_{f(U_n)=y}, (f(U_n), H, U_1)|_{f(U_n)=y})\right] \\
&= \mathop{\mathrm{E}}_{y \leftarrow f(U_n)} [\mathrm{SD}((y, H, H(X_y)), (y, H, U_1))] \\
&\leq \max_{y \in f(\{0,1\}^n)} \mathrm{SD}((y, H, H(X_y)), (y, H, U_1)) \\
&= \max_{y \in f(\{0,1\}^n)} \mathrm{SD}((H, H(X_y)), (H, U_1))
\end{aligned}$$

Proving Lemma 9, cont.

Since $\mathsf{H}_\infty(X_y) = \log(d(n))$ for any $y \in f(\{0,1\}^n)$, the leftover hash lemma (Lemma 6) yields that

$$\begin{aligned} \mathsf{SD}((H, H(X_y)), (H, U_1)) &\leq 2^{(1-\mathsf{H}_\infty(X_y)-2))/2} \\ &= 2^{(1-\log(d(n)))/2} = \mathsf{neg}(n). \quad \square \end{aligned}$$

Section 2

Proving GL – The Computational Case

Proving Goldreich-Levin Theorem

Theorem 11 (Goldreich-Levin)

For $f\colon \{0,1\}^n \mapsto \{0,1\}^n$, *define* $g\colon \{0,1\}^n \times \{0,1\}^n \mapsto \{0,1\}^n \times \{0,1\}^n$ *as* $g(x,r) = (f(x), r)$.

If f *is one-way, then* $b(x,r) := \langle x,r\rangle_2$ *is an hardcore predicate of* g.

Proof: Assume $\exists$ PPT A, $p \in$ poly and infinite set $\mathcal{I} \subseteq \mathbb{N}$ with

$$\Pr[\mathsf{A}(g(U_n,R_n)) = b(U_n,R_n)] \geq \frac{1}{2} + \frac{1}{p(n)}, \qquad (1)$$

for any $n \in \mathcal{I}$, where U_n and R_n are uniformly (and independently) distributed over $\{0,1\}^n$.

We show $\exists$ PPT B and $q \in$ poly with

$$\Pr_{y \leftarrow f(U_n)}[\mathsf{B}(y) \in f^{-1}(y)] \geq \frac{1}{q(n)}, \qquad (2)$$

for every $n \in \mathcal{I}$. In the following fix $n \in \mathcal{I}$.

Focusing on a good set

Claim 12

There exists a set $\mathcal{S} \subseteq \{0,1\}^n$ with

1. $\frac{|\mathcal{S}|}{2^n} \geq \frac{1}{2p(n)}$, and
2. $\Pr\left[\mathsf{A}(f(x), R_n) = b(x, R_n)]\right] \geq \frac{1}{2} + \frac{1}{2p(n)}, \forall x \in S$.

Proof: Let $\mathcal{S} := \{x \in \{0,1\}^n \colon \Pr\left[\mathsf{A}(f(x), R_n) = b(x, R_n)\right] \geq \frac{1}{2} + \frac{1}{2p(n)}\}$.

$$\begin{aligned}\Pr[\mathsf{A}(g(U_n, R_n)) = b(U_n, R_n)] &\leq \Pr[U_n \notin \mathcal{S}] \cdot \left(\frac{1}{2} + \frac{1}{2p(n)}\right) + \Pr[U_n \in \mathcal{S}] \\ &\leq \left(\frac{1}{2} + \frac{1}{2p(n)}\right) + \Pr[U_n \in \mathcal{S}] \Box\end{aligned}$$

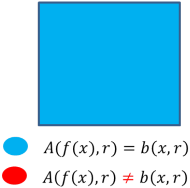
We conclude the theorem's proof showing exist $q \in \text{poly}$ and PPT B:

$$\Pr[\mathsf{B}(f(x)) \in f^{-1}(f(x)) \geq \frac{1}{q(n)}, \qquad (3)$$

for every $x \in \mathcal{S}$. In the following we fix $x \in \mathcal{S}$.

The Perfect Case

$\Pr[\mathsf{A}(f(x), R_n) = b(x, R_n)] = 1$



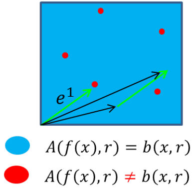
In particular, $\mathsf{A}(f(x), e^i) = b(x, e^i)$ for every $i \in [n]$, where $e^i = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i})$.

Hence, $x_i = \langle x, e^i \rangle_2 = b(x, e^i) = \mathsf{A}(f(x), e^i)$

Algorithm 13 (Inverter B on input y)

Return $(\mathsf{A}(y, e^1), \ldots, \mathsf{A}(y, e^n))$.

Easy case



$$\Pr\left[A(f(x), R_n) = b(x, R_n)\right] \geq 1 - \text{neg}(n)$$

Fact 14

1. $b(x,w) \oplus b(x,y) = b(x, w \oplus y)$ *for every* $w, w, y \in \{0,1\}^n$.
2. $\forall r \in \{0,1\}^n$, *the rv* $(R_n \oplus r)$ *is uniformly distributed over* $\{0,1\}^n$.

Hence, $\forall i \in [n]$:

1. $x_i = b(x, e^i) = b(x,r) \oplus b(x, r \oplus e^i)$ for every $r \in \{0,1\}^n$
2. $\Pr[A(f(x), R_n) = b(x, R_n) \wedge A(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)] \geq 1 - \text{neg}(n)$

Algorithm 15 (Inverter B on input y)

Return $(A(y, R_n) \oplus A(y, R_n \oplus e^1)), \ldots, A(y, R_n) \oplus A(y, R_n \oplus e^n))$.

Proving Fact 14

1. For $w, w, y \in \{0,1\}^n$:

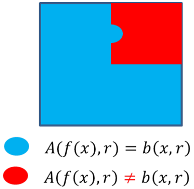
$$\begin{aligned} b(x,y) \oplus b(x,w) &= \left(\bigoplus_{i=1^n} x_i \cdot y_i\right) \oplus \left(\bigoplus_{i=1^n} x_i \cdot w_i\right) \\ &= \bigoplus_{i=1^n} x_i \cdot (y_i \oplus w_i) \\ &= b(x, y \oplus w) \end{aligned}$$

2. For $r, y \in \{0,1\}^n$:

$$\Pr[R_n \oplus r = y] = \Pr[R_n = y \oplus r] = 2^{-n}$$

Intermediate Case

$\Pr[\mathrm{A}(f(x), R_n) = b(x, R_n)] \geq \frac{3}{4} + \frac{1}{q(n)}$



For any $i \in [n]$

$$\begin{aligned}
&\Pr[A(f(x), R_n) \oplus A(f(x), R_n \oplus e^i) = x_i] \\
\geq\ & \Pr[A(f(x), R_n) = b(x, R_n) \wedge A(f(x), R_n \oplus e^i) = b(x, R_n \oplus e^i)] \\
\geq\ & 1 - \left(1 - \left(\frac{3}{4} + \frac{1}{q(n)}\right)\right) - \left(1 - \left(\frac{3}{4} + \frac{1}{q(n)}\right)\right) = \frac{1}{2} + \frac{2}{q(n)}
\end{aligned}$$

Algorithm 16 (Inverter B on input $y \in \{0,1\}^n$)

1. For every $i \in [n]$
   1. Sample $r^1, \ldots, r^v \in \{0,1\}^n$ uniformly at random
   2. Let $m_i = \mathrm{maj}_{j \in [v]}\{(A(y, r^j) \oplus A(y, r^j \oplus e^i)\}$
2. Output $(m_1, \ldots, m_n)$

B's Success Provability

The following claim holds for “large enough" $v = v(n) \in \text{poly}(n)$.

Claim 17

For every $i \in [n]$, it holds that $\Pr[m_i = x_i] \geq 1 - \text{neg}(n)$.

Proof: For $j \in [v]$, let the indicator rv W^j be 1, iff
$A(f(x), r^j) \oplus A(f(x), r^j \oplus e^i) = x_i$.
We want to lowerbound $\Pr\left[\sum_{j=1}^{v} W^j > \frac{v}{2}\right]$.

- The W^j are iids and $\text{E}[W^j] \geq \frac{1}{2} + \frac{2}{q(n)}$ for every $j \in [v]$

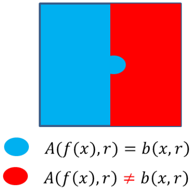
Lemma 18 (Hoeffding's inequality)

Let $X^1, \ldots, X^v$ be iids over $[0,1]$ with expectation μ. Then,
$\Pr\left[\left|\frac{\sum_{j=i}^{v} X^j}{v} - \mu\right| \geq \varepsilon\right] \leq 2 \cdot \exp(-2\varepsilon^2 v)$ *for every* $\varepsilon > 0$.

We complete the proof taking $X^j = W^j$, $\varepsilon = 1/4q(n)$ and $v \in \omega(\log(n) \cdot q(n)^2)$.

The actual (hard) case

$\Pr\left[\mathsf{A}(f(x), R_n) = b(x, R_n)\right] \geq \frac{1}{2} + \frac{1}{q(n)}$



- What goes wrong?

  $\Pr[A(f(x), R_n) \oplus A(f(x), R_n \oplus e^i) = x_i] \geq \frac{2}{q(n)}$
- Hence, using a random guess does better than using A :-<
- Idea: guess the values of $\{b(x, r^1), \ldots, b(x, r^v)\}$
  (instead of calling $\{\mathsf{A}(f(x), r^1), \ldots, \mathsf{A}(f(x), r^v)\}$)

  Problem: negligible success probability

  Solution: choose the samples in a correlated manner

Algorithm B

- Fix $\ell = \ell(n)$ (will be $O(\log n)$) and set $v = 2^\ell - 1$.
- In the following $\mathcal{L} \subseteq [\ell]$ stands for a non empty choice

Algorithm 19 (Inverter B on $y = f(x) \in \{0,1\}^n$)

1. Sample uniformly (and independently) $t^1, \ldots, t^\ell \in \{0,1\}^n$
2. Guess the value of $\{b(x, t^i)\}_{i \in [\ell]}$
3. For all $\mathcal{L} \subseteq [\ell]$: set $r^\mathcal{L} = \bigoplus_{i \in \mathcal{L}} t^i$ and compute $b(x, r^\mathcal{L}) = \bigoplus_{i \in \mathcal{L}} b(x, t^i)$.
4. For all $i \in [n]$, let $m_i = \text{maj}_{\mathcal{L} \subseteq [\ell]}\{\mathsf{A}(f(x), r^\mathcal{L} \oplus e^i) \oplus b(x, r^\mathcal{L})\}$
5. Output $(m_1, \ldots, m_n)$

- Fix $i \in [n]$, and let $W^\mathcal{L}$ be 1 iff $\mathsf{A}(f(x), r^\mathcal{L} \oplus e^i) \oplus b(x, r^\mathcal{L}) = x_i$.
- We want to lowerbound $\Pr\left[\sum_{\mathcal{L} \subseteq [\ell]} W^\mathcal{L} > \frac{v}{2}\right]$
- Problem: the $W^\mathcal{L}$'s are dependent!

Analyzing B's success probability

1. Let $T^1, \ldots, T^\ell$ be iid and uniform over $\{0,1\}^n$.
2. For $\mathcal{L} \subseteq [\ell]$, let $R^{\mathcal{L}} = \bigoplus_{i \in \mathcal{L}} T^i$.

Claim 20

1. $\forall \mathcal{L} \subseteq [\ell]$, $R^{\mathcal{L}}$ is uniformly distributed over $\{0,1\}^n$.
2. $\forall\, w, w' \in \{0,1\}^n$ and $\mathcal{L} \neq \mathcal{L}' \subseteq [\ell]$, it holds that $\Pr[R^{\mathcal{L}} = w \wedge R^{\mathcal{L}'} = w'] = \Pr[R^{\mathcal{L}} = w] \cdot \Pr[R^{\mathcal{L}'} = w'] = 2^{-2n}$.

Proof: (1) is clear, we prove (2) in the next slide.

Proving Fact 20(2)

Assume wlg. that $1 \in (\mathcal{L}' \setminus \mathcal{L})$.

$$\Pr[R^{\mathcal{L}} = w \wedge R^{\mathcal{L}'} = w']$$

$$= \mathop{\mathrm{E}}_{(t^2,\ldots,t^\ell)\leftarrow\{0,1\}^{(\ell-1)n}} \left[\Pr[R^{\mathcal{L}} = w \wedge R^{\mathcal{L}'} = w' \mid (T^2,\ldots,T^\ell) = (t^2,\ldots,t^\ell)]\right]$$

$$= \sum_{(t^2,\ldots,t^\ell):\ (\bigoplus_{i\in\mathcal{L}} t^i)=w} \Pr[(T^2,\ldots,T^\ell) = (t^2,\ldots,t^\ell)]$$

$$\cdot \Pr[R^{\mathcal{L}'} = w' \mid (T^2,\ldots,T^\ell) = (t^2,\ldots,t^\ell)]$$

$$= \sum_{(t^2,\ldots,t^\ell):\ (\bigoplus_{i\in\mathcal{L}} t^i)=w} \Pr[(T^2,\ldots,T^\ell) = (t^2,\ldots,t^\ell)] \cdot 2^{-n}$$

$$= 2^{-n} \cdot 2^{-n} = \Pr[R^{\mathcal{L}} = w] \cdot \Pr[R^{\mathcal{L}'} = w'].$$

□

Pairwise independence variables

Definition 21 (pairwise independent random variables)

A sequence of random variables $X^1, \ldots, X^v$ is pairwise independent, if $\forall i \neq j \in [v]$ and $\forall a, b$, it holds that

$$\Pr[X^i = a \wedge X^j = b] = \Pr[X^i = a] \cdot \Pr[X^j = b]$$

- By Claim 20, $r^{\mathcal{L}}$ and $r^{\mathcal{L}'}$ (chosen by B) are pairwise independent for every $\mathcal{L} \neq \mathcal{L}' \subseteq [\ell]$.
- Hence, also $W^{\mathcal{L}}$ and $W^{\mathcal{L}'}$ are.
  (Recall, $W^{\mathcal{L}}$ is 1 iff $\mathsf{A}(f(x), r^{\mathcal{L}} \oplus e^i) \oplus b(x, r^{\mathcal{L}}) = x_i$)

Lemma 22 (Chebyshev's inequality)

Let $X^1, \ldots, X^v$ be pairwise-independent random variables with expectation μ and variance σ^2. Then, for every $\varepsilon > 0$,

$$\Pr\left[\left|\frac{\sum_{j=1}^{v} X^j}{v} - \mu\right| \geq \varepsilon\right] \leq \frac{\sigma^2}{\varepsilon^2 v}$$

B's success provability, cont.

Assuming that B always guesses $\{b(x, t^i)\}$ correctly, then for every $\mathcal{L} \subseteq [\ell]$

- $\mathrm{E}[W^{\mathcal{L}}] \geq \frac{1}{2} + \frac{1}{q(n)}$
- $\mathrm{Var}(W^{\mathcal{L}}) := \mathrm{E}[W^{\mathcal{L}}]^2 - \mathrm{E}[(W^{\mathcal{L}})^2] \leq 1$

Taking $\varepsilon = 1/2q(n)$ and $v = 2n/\varepsilon^2$ (i.e., $\ell = \lceil \log(2n/\varepsilon^2) \rceil$), Lemma 22 yields that

$$\Pr[m_i = x_i] = \Pr\left[\frac{\sum_{\mathcal{L} \subseteq [\ell]} W^{\mathcal{L}}}{v} > \frac{1}{2}\right] \geq 1 - \frac{1}{2n} \qquad (4)$$

Hence, by a union bound, B outputs x with probability $\frac{1}{2}$.

Taking the guessing into account, yields that B outputs x with probability at least $2^{-\ell}/2 \in \Omega(n/q(n)^2)$.

Reflections

- Hardcore functions:

  Similar ideas allows to output $\log n$ "pseudorandom bits"

- Alternative proof for the LHL:

  Let X be a rv with over $\{0,1\}^n$ with $H_\infty(X) \geq t$, and assume $SD((R_n, \langle R_n, X\rangle_2), (R_n, U_1)) > \alpha = 2^{-c \cdot t}$ for some universal $c > 0$.

$\implies$ Exists (a possibly inefficient) algorithm D that distinguishes $(R_n, \langle R_n, X\rangle_2)$ from (R_n, U_1) with advantage α

$\implies$ Exists algorithm A that predicts $\langle R_n, X\rangle_2$ given R_n with prob $\frac{1}{2} + \alpha$

$\implies$ (by GL) Exists algorithm B that guesses X from nothing, with prob $\alpha^{O(1)} > 2^{-t}$

Reflections cont.

- List decoding:

  An encoder $C\colon \{0,1\}^n \mapsto \{0,1\}^m$ and a decoder D, such that the following holds for any $x \in \{0,1\}^n$ and c of hamming distance $\frac{1}{2} - \delta$ from $C(x)$:

  $D(c, \delta)$ outputs a list of size at most $\text{poly}(1/\delta)$ that whp. contains x

  The code we used here is known as the Hadamard code

- LPN - learning parity with noise:

  Find x given polynomially many samples of $\langle x, R_n \rangle_2 + N$, where $\Pr[N = 1] \leq \frac{1}{2} - \delta$.

  The difference comparing to Goldreich-Levin – no control over the R_n's.