Computational Models — Lecture 11

Handout Mode

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1 Based on frames by Benny Chor, Tel Aviv University, modifying frames by Maurice Herlihy, Brown University.
Ladies and Gentlemen, Boys and Girls

We are about to begin the fourth part of the course:

Introduction to Computational Complexity
Talk Outline

- The class $\mathcal{P}$ (continue from last week)
- The class $\mathcal{NP}$
- Verifiability
- Additional NP languages
- The class $co-\mathcal{NP}$
- $\mathcal{P}$ Verses $\mathcal{NP}$
- NP-completeness

Sipser's book 7.1–7.4
Part I

Introduction to Time Complexity
How long does it take to decide \( L = \{0^n1^n : n \geq 0\} \)

Clearly \( L \in \mathcal{R} \).

**Question 1**

How much time does a single-tape TM need to decide \( L \)?

**Algorithm 2 (Decider \( M_1 \) for \( L \))**

On input string \( w \):

1. Scan across tape and **Reject** if 0 is found to the right of a 1.
2. While both 0s and 1s appear on tape, repeat the following:
   - Scan across tape, crossing of a single 0 and a single 1 in each pass.
3. **Accept** if no 0s and 1s remain; otherwise **Reject**.

We analyze the time it takes to perform each of the three steps separately. In the following let \( n = |w| \).
Analyzing Step 1

Scan across tape and Reject if 0 is found to the right of a 1. If not, return to starting point.

- Scanning requires $n$ steps.
- Re-positioning head requires $n$ steps.
- Total is $2n = O(n)$ steps.
Analyzing Step 2

While both 0s and 1s appear on tape, repeat the following:

*Scan across tape, crossing of a single 0 and a single 1 in each pass.*

- Each scan requires $O(n)$ steps.
- Since each scan crosses off two symbols, the number of scans is at most $n/2$.
- Total number of steps is $(n/2) \cdot O(n) = O(n^2)$. 
Analyzing Step 3

If 0s still remain after all 1s have been crossed out, or vice-versa, Reject. Otherwise, if the tape is empty, Accept.

- Single scan requires $O(n)$ steps.
- Total is $O(n)$ steps.
Overall cost

Total cost for three steps

1. $O(n)$
2. $O(n^2)$
3. $O(n)$

which is $O(n^2)$
**Deterministic Time**

**Definition 3 (deterministic Time)**

Let $M$ be a deterministic TM, and let $t : \mathbb{N} \mapsto \mathbb{N}$. We say that $M$ runs in time $t(n)$, if for every input $x$ of length $n$, the number of steps that $M(x)$ uses is at most $t(n)$.

**Question 4**

What is a “step"?

**Definition 5 (DTIME)**

For $t : \mathbb{N} \mapsto \mathbb{N}$, let $\text{DTIME}(t(n)) =$

\[ \{ L \subseteq \Sigma^* : \text{L is decided by an } O(t(n))-\text{time single tape TM} \} \]

Note that $t(n)$ running time, is also required for strings not in $L$. 
Let $L = \{1^n : n \in \mathbb{N}\}$. (over $\Sigma$)

Is $L$ in $\text{DTIME}(n)$? in $\text{DTIME}(n^{1/2})$? Proof?

Let $\text{PATH} = \{\langle G, s, t \rangle : G \text{ has directed path from } s \text{ to } t\}$

Is $\text{PATH}$ in $\text{DTIME}(n)$? in $\text{DTIME}(n^{1/2})$?

The questions are not well defined w/o defining the encoding of strings into triplets $\langle G, s, t \rangle$.

Encoding may matter much — an exponential algorithm with respect to one encoding might turn to linear algorithm with respect to other encoding.

If not defined explicitly, we assume a “reasonable encoding”:

- Graphs encoding: adjacency list or adjacency matrix
- Integers encoding: binary encoding (and not unary)
Factorization algorithm

1. Do for $i = 2$ to $q - 1$:
   - If $i$ divides $q$ output $i$ and halt.
2. Output "none" and halt.

What is the running time?

- $O(q)$ (easily improved to $O(\sqrt{q})$)
- Yay: poly time! ? Yes, if $q$ represented in unary.
- Uh oh... number of bits to represent $q$ in binary is $\log q$, so $O(q)$ is EXPONENTIAL IN THE INPUT LENGTH

Today’s crypto systems assume that factoring 4,000 bit numbers takes a long time!
Relations among time classes

Let $t_1, t_2 : \mathbb{N} \rightarrow \mathbb{N}$ be two functions.

**Claim 6**

If $t_1(n) = O(t_2(n))$, then $\text{DTIME}(t_1(n)) \subseteq \text{DTIME}(t_2(n))$.

Stated informally, more time does not hurt. But does it actually help?

Would like to say “if $t_1(n) = o(t_2(n))$ then $\text{DTIME}(t_1(n)) \subsetneq \text{DTIME}(t_2(n))$”.

But this is what we can get:

**Claim 7**

(Assume that $t_1(n)$ and $t_2(n)$ are time constructible\(^a\). If $t_1(n) \in O(t_2(n)/\log(n))$, then $\text{DTIME}(t_1(n)) \subsetneq \text{DTIME}(t_2(n))$.

\(^a\) $t$ is time constructible if the function mapping $1^n$ to the binary representation of $t(n)$ is computable in time $O(t(n))$.

Informally, sufficiently more time does help

Proof omitted (take complexity course, or search for “time hierarchy theorem”)

We have seen that $L \in \text{DTIME}(n^2)$. Can we do faster?

**Algorithm 8 (Decider $M_2$ for $L$)**

On input string $w$

1. Scan across tape and **Reject** if 0 is found to the right of a 1.

2. Repeat the following while both 0s and 1s appear on tape:
   
   2.1 Scan across tape, checking whether total number of 0s plus 1s is even or odd. If odd, **Reject**.

   2.2 Scan across tape, crossing off every other 0 (starting with the first), and every other 1 (starting with the first) in each pass.

3. **Accept** if all string is crossed off; Otherwise **Reject**.

**Example:** $w = 0^{13}1^{13}$
Correctness?

- Let binary representation of number of 0’s and 1’s be $a_k \ldots a_0$ and $b_\ell \ldots b_0$ respectively.
- If total number of 0s plus 1s is even, they have same parity $\implies a_0 = b_0$.
- Crossing off every other 0 and 1 has same effect as “shift right” on binary representation.
- So, next iteration checks $a_1 = b_1$...
- When finish, know that $a_i = b_i$ for all $i$!
Running time analysis of $M_2$

Algorithm 9 (Decider $M_2$ for $L$)

On input string $w$

1. Scan across tape and \textbf{Reject} if 0 is found to the right of a 1.

2. Repeat the following while both 0s and 1s appear on tape:
   
   2.1 Scan across tape, checking whether total number of 0s plus 1s is even or odd. If odd, \textbf{Reject}.

   2.2 Scan across tape, crossing off every other 0 (starting with the first), and every other 1 (starting with the first) in each pass.

3. \textbf{Accept} if all string is crossed off; Otherwise \textbf{Reject}.

- One pass in each step (1,2,3) takes $O(n)$ time.
- \textbf{Steps 1,3}: each executed once
- \textbf{Step 2} executed $1 + \log_2 n$ times
- Total for \textbf{Step 2} is $(1 + \log_2 n)O(n) = O(n \log n)$.
- Grand total: $O(n) + O(n \log n) = O(n \log n)$. 
Further improvements, anybody?

**Question 10**
Can the running time be made $o(n \log n)$?

**Answer:** Not on a single tape TM.

**Claim 11**
$L \notin \text{DTIME}(o(n \log n))$ (i.e., $L \notin \text{DTIME}(g(n))$ for any $g(n) \in o(n \log n)$).

Hence, $CFL \notin \text{DTIME}(o(n \log n))$.

In contrast, regular languages $\subseteq \text{DTIME}(O(n))$. (?)

The proof of the claim is an immediate corollary of the following lemma.

**Lemma 12 (pumping lemma for low-time TMs)**
For every TM $M$ of running time $t(n) \in o(n \log n)$, exists $\ell \in \mathbb{N}$ such that:

- every $w \in L = L(M)$ with $|w| \geq \ell$, can be written as $w = xyz$, $|y| > 0$ and $xy^i z \in L$ for every $i \geq 0$.

We only prove for $i = 2$. 
Proving the pumping lemma

Fix an $o(n \log n)$-time TM $M = (Q, \Sigma, \cdot, \cdot, \cdot, \cdot, \cdot)$ and let $L = L(M)$.

**Definition 13 (Crossing sequence)**

The crossing sequence of $M(w)$ at location $i$, is the sequence of states used by the invocation of $M$ on input $w$, when moving from $i$ to $(i + 1)$, and from $(i + 1)$ to $i$. (i.e., the state when applying $\delta$ in each move).

**Claim 14 (Identical crossing sequences)**

$\exists \ell \in \mathbb{N}$ s.t. for $w \in \{0, 1\}^*$ with $|w| \geq \ell$, exists $i \neq i' \in [|w|]$, s.t. the crossing sequence of $M(w)$ at $i$ and $i'$ is the same.

**Claim 15 (Identical crossing sequences yields pumping)**

Let $w = xyz$, and assume that in $M(w)$ has identical crossing sequences at $|x|$ and $|x + y|$, then $M(w)$ and $M(w' = (x, y^2, z))$ halt in the same state.

- Let $w \in L$ be of length at least $\ell$.
- By Claim 14, $w$ can be written as $w = xyz$ ($y \neq \varepsilon$) such that the crossing sequence of $M(w)$ at locations $|x|$ and $|xy|$ is the same.
- By Claim 15, $xy^2z \in L$. 
Proof of Claim 14

Claim 16 (Restating Claim 14)

\[ \exists \ell \in \mathbb{N} \text{ s.t. for } w \in \{0, 1\}^* \text{ with } |w| \geq \ell, \text{ exists } i \neq i' \in [|w|], \text{ s.t. the crossing sequence of } M(w) \text{ at } i \text{ and } i' \text{ is the same.} \]

Consider the run \( M(w) \) for some fix \( w \in \{0, 1\}^n \).

- Let \( C \) be subset of the first \( n \) locations on tape visited at most \( f(n) := 2 \cdot t(n)/n \) (at most twice the average).
- Crossing sequence length at location \( i \in C \) is at most \( f(n) \).
- Let \( C' \) be subset of the first \( n \) locations on tape visited strictly greater than \( f(n) \) times.
- By a simple average argument, \( |C'| \leq n/2 \).
- Since \( |C| + |C'| = n \), it holds that \( |C| \geq n/2 \).
- \# of different crossing sequences in \( C \) is at most \( (|Q| + 1)^{f(n)} = 2^{f(n) \log(|Q|+1)} \).
- Let \( \ell \) be such \( f(n) \cdot \log(|Q| + 1) < \log n - 1 \) for every \( n \geq \ell \).
- If \( n \geq \ell \), then \# of different crossing sequences in \( C \) is smaller than \( n/2 \).
- Hence, exists pair \( i, j \in C \) with the same crossing sequence. \( \square \)
Proof of Claim 15, idea

2 identical crossing sequences
Proof of Claim 15

Claim 17 (Restating Claim 15)
Let \( w = xyz \), and assume that in \( M(w) \) has identical crossing sequences at \(|x|\) and \(|x + y|\), then \( M(w) \) and \( M(w' = (x, y^2, z)) \) halt in the same state.

The augmented (partial) configuration at locations \([i, k]\), is the content of the cells in these locations, the head location and state if the head is in these locations, and the crossing sequences at location \(i - 1\) and \(k\).

Let \( \text{conf}_{z,t}(i, k) \) be the augmented configuration of \( M(z) \) after \( t \) steps at \([i, k]\).

Claim 18

For any \( t \in \mathbb{N} \), it holds that

- \( \exists t_1 \) such \( \text{conf}_{w',t}(1, |x + y|) = \text{conf}_{w,t}(1, |x + y|) \)
- \( \exists t_2 \) such \( \text{conf}_{w',t}(|x + y| + 1, \infty) = \text{conf}_{w,t}(|x| + 1, \infty) \)

Proving Claim 15 using Claim 18: (fill the details)

- Prove that \( M(w') \) halts.
- Prove that \( M(w') \) accepts.
Proof sketch for Claim 18

Claim 19 (Restating Claim 18)

For any \( t \in \mathbb{N} \), it holds that

1. \( \exists t_1 \) such \( \conf_{w',t}(1, |x + y|) = \conf_{w,t_1}(1, |x + y|) \)
2. \( \exists t_2 \) such \( \conf_{w',t}(|x + y| + 1, \infty) = \conf_{w,t_2}(|x| + 1, \infty) \)

Proof by induction on \( t \). Only challenging part is when at step \( t \) head crosses \((|x + y|, |x + y| + 1)\).

Assume after step \( t - 1 \) head of \( M(w') \) is at location \( |x + y| \), and that after step \( t \) head in location \( |x + y| + 1 \).

Let \( q_1, \ldots, q_\ell \) is the crossing sequence at \( |x + y| \) (after step \( t \)).

1. Let \( t_1 \) and \( t_2 \) be the value guaranteed by the .i.h. for \( t - 1 \).
2. It is clear that \( \conf_{w',t}(1, |x + y|) = \conf_{w,t_{1+1}}(1, |x + y|) \).
3. Head location in not in \( \conf_{w,t_2}(|x| + 1, \infty) \).
4. By 2, at some step \( t'_2 > t_2 \) the head in \( M(w) \) crosses \((|x|, |x| + 1)\) to generate crossing sequence \( q_1, \ldots, q_\ell \).
5. Hence, \( \conf_{w',t}(|x + y| + 1, \infty) = \conf_{w,t'_2}(|x| + 1, \infty) \)
A two-tape decider for $L = \{0^n1^n : n \geq 0\}$

Algorithm 20 (Two-tape Decider $M_3$ for $L$)

On input string $w$:

1. Scan across tape and Reject if 0 is found to the right of a 1.
2. Scan across 0s to first 1, copying 0s to tape 2.
3. Scan across 1s on tape 1 until the end. For each 1, cross off a 0. If no 0s left, Reject.
4. If any 0s left, Reject; otherwise Accept.

Question 21
What is $M_3$ running time?
Complexity of deciding \( L = \{0^n1^n\} \)

- Single-tape \( M_1: O(n^2) \).
- Single-tape \( M_2: O(n \log n) \) (fastest possible!).

Hence \( L \in \text{DTIME}(O(n \log n)) \), but not in \( \text{DTIME}(O(f(n))) \) for \( f(n) \in o(n \log n) \)

- Two-tape \( M_3: O(n) \).

Important difference between complexity and computability:

- Computability: all reasonable models equivalent (Church-Turing)
- Complexity: choice of model does affect running time.

**Question 22**

By how much does a model affect complexity?
Multitape speedup

Let $t(n)$ be a function where , and let $L \subseteq \Sigma^*$ be a language.

**Claim 23**

Assume $\exists t(n)$-time multitape TM that decides $L$ with $t(n) \geq n$, then $\exists$ an $O(t(n)^2)$-time single-tape TM that decides $L$. 

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Computational Models, Lecture 11  
January 01, 2018  
25 / 74
Reminder: Emulating multitape TMs

Algorithm 24 (Single tape emulator $S$ for $k$-tape $M$)

On input $w = w_1 \cdots w_n$:

1. Write $\# w_1 w_2 \cdots w_n \# \# \# \# \cdots \#$ on tape.
2. Scan tape from first $\#$ to $(k + 1)$-st $\#$ to read symbols under virtual heads.
3. Rescan tape to write new symbols and move heads.
4. If need to move virtual head onto $\#$, shift tape content to the right.

- For each step of $M$, the emulator $S$ performs 2 scans and up to $k$ rightward shifts
- On input of length $n$, $M$ makes $O(t(n))$ many steps, so active portion of each tape is $O(t(n))$ long.
- Total number of steps $S$ makes:
  - $O(k \cdot t(n)) = O(t(n))$ steps to simulate one step of $M$.
  - Initial tape arrangement $O(n)$.
  - Grand total: $O(n) + O(t(n)^2) = O(t(n)^2)$ steps (recall $t(n) > n$).
Non-deterministic time

**Definition 25 (nondeterministic time)**

A non-deterministic TM $N$ runs in time $f(n)$, where $f : \mathbb{N} \rightarrow \mathbb{N}$, if for every input $x$ of length $n$, the maximum number of steps that $N$ uses on any branch of its computation tree on $x$, is at most $f(n)$.

Notice that also non-accepting branches must reject within $f(n)$ many steps.

**TAKE NOTE:** the depth of the tree, not the size of the tree!!!
Non-determinism speedup

Claim 26

Suppose $N$ is a nondeterministic TM that runs in time $t(n)$ and decides the language $L$. Then there exists a $2^O(t(n))$-time deterministic TM $D$ that decided $L$.

Note contrast with multi-tape result.
Proof’s idea: $D$ emulates $N$. Reminder

- $D$ tries all possible branches (say using BFS).
  - Accept if finds any accepting state.
  - Reject if all branches reject.
- Since $N$ always stops, exactly one of two possibilities must occur.
Emulation details

We view the computation of $N$ as a tree, whose nodes are configurations of the TM (i.e., state, head location and tape content).

- Root is starting configuration,
- Fanout is at most some constant $b$ (??),
- Depth at most $\leq t(n)$,
- Total number of nodes $\mathcal{O}(b^{t(n)})$,
- Emulation time is $\mathcal{O}(b^{t(n)}) = \mathcal{O}(2^{\mathcal{O}(t(n))})$
Remarks

1. Breadth-first search used in emulation
   - Visit each node.
   - May be improved upon by using depth-first search (is it OK?) or other tree search strategies.
   - Still, doing this may save constants, but nothing substantial (?)

2. Simulation uses a three-tape machine.

Single-tape simulation: \((2^{O(t(n))})^2 = 2^{O(2t(n))} = 2^{O(t(n))}\).
Polynomial is Good, exponential is Bad

Assume one step takes $1/10^6$ fraction of a second....

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Exponential time factoring algorithms

2008 claim on forum:

1 hr for 120 bits
10 hrs for 150 bits
100 hrs for 180 bits
1000 hrs for 210 bits
Gaps between models

- At most polynomial gap in time to perform tasks between different deterministic models (single- vs. multi-tape TMs, TM vs. RAM, etc.)
- Apparently exponential gap in time to perform tasks between deterministic and non-deterministic models.

Claim 27
All “reasonable” classical (not including quantum) models of computation are polynomially equivalent.

Any one can simulate another with only polynomial blowup in running time.

Fact 28
Is a given problem solvable in
- Linear time? model-specific.
- Polynomial time? model-independent.
Part II

The Class P
The class $\mathcal{P}$

$\mathcal{P}$ is the set of languages decidable in polynomial time on deterministic TMs.

**Definition 29 ($\mathcal{P}$)**

$$\mathcal{P} = \bigcup_{c \geq 0} \text{DTIME}(n^c)$$

The class $\mathcal{P}$ is important because:

- Invariant for all (deterministic) models of computation polynomially equivalent to TMs
  - not affected by particulars of model . . .
  - go ahead, have another tape, they’re pretty small and inexpensive . . .
- Roughly corresponds to *realistically solvable* (tractable) problems.
  - actually depends on context
  - going from exponential to polynomial algorithm usually requires major insight,
  - if you find an inefficient polynomial algorithm, you can often find a more efficient one.
Known problems in $\mathcal{P}$

- **Integer arithmetic**: Addition, subtraction, multiplication, division with remainder.
- **Modular arithmetic**: Exponentiation (RSA), inverse.
- **Integer Algorithms**: Greatest common divisor (gcd).
- **Operations research**: Maximum network flow, linear programming.
- **Algebra**: Matrix multiplication, computing determinants, matrix inversion, solving systems of linear equations, factoring polynomials.
- **Graph algorithms**: BFS and DFS in graphs, minimum spanning trees, finding Eulerian path.
Theorem 30

\[ \text{PATH} \in \mathcal{P}. \]

Algorithm 31 (\(M_1\) – Naive algorithm for PATH)

Input: an \(m\) nodes graph \(G\)
For each path \(p\) in \(G\) of length \(\leq m\) : check if \(p\) goes from \(s\) to \(t\).

Question 32

What is the (time) complexity of \(M_1\)?

- there are \(m^m\) possible paths
  \[ \rightarrow \text{ exponential in number of nodes} \]
  \[ \rightarrow \text{ exponential in input size} \]
- Oh, oh. Does not sound like \(\mathcal{P}\) to me . . .
Efficient algorithm for \( \text{PATH} \)

**Algorithm 33 (\( M_2 \) – efficient algorithm for \( \text{PATH} \))**

1. Place mark on \( s \).

2. Repeat until no additional nodes marked:
   - **2.1** Scan edges of \( G \).
   - **2.2** If edge \((a, b)\) found from marked node \( a \) to unmarked node \( b \), mark node \( b \).

3. **Accept** if \( t \) is marked; otherwise **Reject**.

**Question 34**

What is the complexity of \( M_2 \)?

- Steps 1 and 3 run once.
- Step 2 runs at most \( m \) times, because each time (except last) it marks at least one new node.

\[ \Rightarrow \text{total number of steps is polynomial.} \]
Relative primality

Two numbers are relatively prime if their greater common divisor (gcd) is 1 (i.e., the largest integer that divides them both).

\[
\begin{align*}
gcd(10, 21) &= 1 \implies 10 \text{ and } 21 \text{ are relatively prime} \\
gcd(10, 22) &= 2 \implies 10 \text{ and } 22 \text{ are not relatively prime}
\end{align*}
\]

Definition 35

\[\text{RELPRIME} = \{\langle x, y \rangle : \gcd(x, y) = 1\}.\]

Theorem 36

\[\text{RELPRIME} \in \mathcal{P}.\]
Naive algorithm for RELPRIME

Algorithm 37 (Naive algorithm for RELPRIME)

Input: integers $x, y$:
Search through all possible divisors of $x, y$ and test divisibility.

- If $x, y$ are give in unary:
  - Size of $\langle x \rangle$ is $x$
  - Testing all potential divisors of $x, y$ is polynomial

- If $x, y$ are give in binary
  - Size of $\langle x \rangle$ is $\log x$
  - Testing all potential divisors of $x, y$ is exponential

- To analyze the running time of an algorithm that decides a language $L$, one should first say what is the encoding of $L$.

- Yet, we sometimes ignore that, and assume “reasonable encoding”

- The above algorithm is sometimes called pseudo polynomial.
Euclid’s Algorithm for Computing \( \gcd \)

**Algorithm 38 \((E)\)**

On input \( \langle x, y \rangle \):

1. Repeat until \( x = 0 \):
   1.1 If \( y > x \), swap \( x \) and \( y \).
   1.2 \( x \leftarrow x \mod y \)
2. Output \( y \).

**Claim 39**

\( E \) runs in polynomial-time and correctly computes the \( \gcd \) function.

**Algorithm 40 \((M\text{ for RELPRIME})\)**

On input \( \langle x, y \rangle \):

Accept iff \( E(x, y) = 1 \).

To prove that \( \text{RELPRIME} \in \mathcal{P} \), we only need to prove **Claim 39**.
Analyzing Euclid’s algorithm

We will only prove the running time part.

**Algorithm 41 (E)**

On input \( \langle x, y \rangle \):

1. Repeat until \( x = 0 \):
   1.1 If \( y > x \), swap \( x \) and \( y \).
   1.2 \( x \leftarrow x \mod y \)
2. Output \( y \).

- Each execution of Step 1.1 cuts \( x \) by at least half (case analysis for \( y < x/2 \) and \( x/2 \leq y < x \))
- After each two executions maximal value is cut in half\
  \[ \Rightarrow \text{number of stages is } \min(\log_2 x, \log_2 y) \Rightarrow \text{total running time is polynomial.} \]

Consequently, \( \text{RELPRIME} \in \mathcal{P} \).
Section 1

The Class NP
The Class $\mathcal{NP}$

**Definition 42 (NTIME)**

For $t: \mathbb{N} \mapsto \mathbb{N}$, let $\text{NTIME}(t(n)) = \{ L \subseteq \Sigma^* : L \text{ is decided by an } O(t(n))-\text{time single tape NTM} \}$

$\mathcal{NP}$ is the set of languages decidable in polynomial time on non-deterministic TMs.

**Definition 43 ($\mathcal{NP}$)**

$\mathcal{NP} = \bigcup_{c \geq 0} \text{NTIME}(n^c)$

The class $\mathcal{NP}$ is important because:

- Insensitive to choice of reasonable non-deterministic computational model.
- Roughly corresponds to problems whose positive solutions are efficiently verified.
Are language in $\mathcal{NP}$ decidable in polynomial time?

We don’t know!

We do know is this:

Large class of fundamental languages in $\mathcal{NP}$ that are “the hardest”
(i.e., if ONE is efficiently solvable then ALL are efficiently solvable)
A Hamiltonian path in a directed $G$ visits each node exactly once.

$$\text{HAMPATH} = \{ \langle G, s, t \rangle : G \text{ has Hamiltonian path from } s \text{ to } t \}$$

**Question 44**

How hard is it to decide HAMPATH?

Easy to obtain exponential time algorithm:

- Generate each potential path
- Check whether it is Hamiltonian
HAMPATH $\in \mathcal{NP}$

Here is an NTM that decides HAMPATH in polynomial time.

**Algorithm 45 ($N$)**

On input $\langle G = (V, E), s, t \rangle$,

1. **Guess** a list of numbers $p_1, \ldots, p_m$, where $m = |V|$ and $1 \leq p_i \leq m$.

2. **Accept** if all the following hold (otherwise **Reject**):
   - No repetitions in list
   - $p_1 = s$ and $p_m = t$.
   - $(p_i, p_{i+1}) \in E$ for every $1 \leq i \leq m - 1$

- How does a TM guess a string?

**Claim 46**

$N$ runs in polynomial time
Verifiability of HAMPATH

This problem has one very interesting feature: polynomial verifiability:

We don’t know a fast way to find a Hamiltonian path, but we can check whether a given path is Hamiltonian in polynomial time.

Verifying correctness of a path is much easier than determining whether one exists
Section 2

Verifiability
Verifiability

Definition 47 (verifier)

A \textit{deterministic} algorithm \( V \) is a \textit{verifier} for a language \( L \), if

\begin{itemize}
  \item \( x \in L \implies \exists c \in \{0, 1\}^* \text{ s.t. } V(x, c) = 1. \)
  \item \( x \notin L \implies \nexists c \in \Sigma^* \text{ s.t. } V(x, c) = 1. \)
\end{itemize}

\begin{itemize}
  \item The verifier uses the additional information \( c \) to verify \( x \in L \).
  \item If \( V \) accepts \((x, c)\) (i.e., outputs 1), the string \( c \) is called a \textit{certificate} (also known as, \textit{proof} or \textit{witness}) for \( x \).
  \item A \textit{polynomial verifier} runs in polynomial time in \(|x|\) (i.e., in the length of its left-hand-side input parameter).
  \item A language \( L \) is \textit{polynomially verifiable}, if it has a polynomial verifier.
  \item A certificate for \( \langle G, s, t \rangle \in \text{HAMPATH} \) is simply the Hamiltonian path from \( s \) to \( t \).
    Easy to \textit{verify} in time polynomial in \(|\langle G, s, t \rangle|\) whether given path is Hamiltonian.
  \item \textbf{Not} all languages are known to be polynomially verifiable.
\end{itemize}
NP and Verifiability

Theorem 48

A language is in $\mathcal{NP}$ iff it has a polynomial time verifier.

Proof’s idea:

- The NTM emulates the verifier by guessing the certificate.
- Verifier emulates NTM by using accepting branch as certificate.
Verifiability $\implies \mathcal{NP}$

**Claim 49**

If $L$ has a poly-time verifier, then it is decided by some polynomial-time NTM.

**Proof:** Let $V$ be poly-time verifier for $L$ of running time $p(n)$ for some $p \in \text{poly}$.

**Algorithm 50 ($N$)**

On input $x \in \{0, 1\}^n$:

1. **Guess** a string $c$ of length $p(n)$.
2. Emulate $V$ on $\langle x, c \rangle$
3. **Accept** if $V$ accepts; Otherwise **Reject**.

Why is it suffices to guess a string of length $p(n)$?
Claim 51

If $L$ is decided by a polynomial-time NTM $N$, then $L$ has a poly-time verifier.

Proof: Assume for simplicity that at each step of $N$, the number of possible non-deterministic moves is at most two.

Algorithm 52 ($V$)

On input $(x, c)$:

1. Emulate $N(x)$, treating each symbol of $c$ as a description of the non-deterministic choice in each step of $N$.
2. **Accept** if this branch accepts; Otherwise **Reject**.

Without the simplifying assumption?
Section 3

A few more NP languages
A clique in a graph is a subgraph where every two nodes are connected by an edge.

A \textit{k-clique} is a clique of size \textit{k}.

\textbf{Question 53}

What is the largest \textit{k}-clique in the figure?
CLIQUE cont.

CLIQUE = \{⟨G, k⟩: G is an undirected graph with a k-clique\}

**Theorem 54**

\[
\text{CLIQUE} \in \mathcal{NP}
\]

Proof’s idea: The clique is the certificate.

**Algorithm 55 (V)**

On input \(⟨G, k⟩, c⟩

Accept if \(c\) is a \(k\)-clique subgraph of \(G\);
Otherwise Reject.
An independent set in a graph is a set of vertexes, no two of which are linked by an edge.

A \( k \)-IS is an independent set of size \( k \).

**Question 56**

What is the largest \( k \)-IS in the figure?
Independent set cont.

\[ \text{IND-SET} = \{ \langle G, k \rangle : G \text{ contains an independent set of size } k \} \]

**Theorem 57**

\[ \text{IND-SET} \in \mathcal{NP} \]

Proof’s idea: The independent set is the certificate.

**Algorithm 58 (V)**

On input \( \langle G, k \rangle, c \)

Accept if \( c \) is a \( k \)-IS of \( G \) (no edges between nodes in \( c \), and \( |c| = k \)); Otherwise Reject.
Section 4

co-$\mathcal{NP}$
The class $\text{co-NP}$

$\overline{\text{CLIQUE}} = \{ \langle G, k \rangle : G \text{ is an undirected graph with no } k\text{-clique} \}$ seems not to be member of $\text{NP}$.

It seems harder to efficiently verify that something does not exist than to efficiently verify that something does exist.

**Definition 59 ($\text{co-NP}$)**

$\text{co-NP} = \{ L : \overline{L} \in \text{NP} \}$.

But.. we are not sure...So far, no one knows if $\text{co-NP}$ is distinct from $\text{NP}$.

**Claim 60**

$\mathcal{P} \subseteq \text{co-NP}$.

Proof? $L \in \mathcal{P} \implies \overline{L} \in \mathcal{P} \implies \overline{L} \in \text{NP} \implies L \in \text{co-NP}$.

Is Primality in $\text{NP}$? $\text{co-NP}$?

How would you prove that a number is prime without trying all divisors? Actually it is in P! (not obvious at all)
Section 5

P vs. NP
The question $\mathcal{P} \overset{?}{=} \mathcal{NP}$ is one of the great unsolved mysteries in contemporary mathematics.
Most computer scientists believe the two classes are not equal
Most bogus proofs show them equal (?)
One of 7 Clay Millenium Prize problems (1,000,000$!)
“Computer Science’s greatest intellectual export” (Papadimitriou 2007)
\( P \) Vs. \( NP \), cont.

If \( P \) differs from \( NP \), then the distinction between \( P \) and \( NP \setminus P \) is meaningful and important.

- Languages in \( P \) are \textit{tractable}
- Languages in \( NP \setminus P \) are \textit{intractable}

Until we can prove that \( P \neq NP \), there is no hope of proving that a specific language lies in \( NP \setminus P \).

Nevertheless, we can prove statements of the form

\[
\text{If } A \in NP \setminus P, \text{ then } B \in NP \setminus P.
\]
Section 6

NP Completeness
NP Completeness

The class of NP-complete languages are

- “hardest” languages in \( \mathcal{NP} \)
- If any NP-complete \( L \in P \), then \( \mathcal{NP} = \mathcal{P} \).

Question 61
Are there NP-complete languages?
### Definition 62 (poly-time computable functions)

A function \( f : \Sigma^* \mapsto \Sigma^* \) is polynomial-time computable, if there is a poly-time deterministic TM that
- starts with input \( w \), and
- halts with \( f(w) \) on tape.

### Definition 63 (poly-time reduction)

A polynomial-time computable \( f : \Sigma^* \mapsto \Sigma^* \) is a poly-time reduction from language \( A \) to \( B \), if \( x \in A \iff f(x) \in B \) for every \( x \in \Sigma^* \).

Is such a reduction from \( A \) to \( B \) exists, we say that \( A \) is poly-time mapping reducible to \( B \), denoted \( A \leq_P B \).

The mapping \( f \) efficiently converts questions about membership in \( A \) to membership in \( B \).
Example: CLIQUE $\leq_P$ IND-SET

Proof:

**Definition 64**

The complement of a graph $G = (V, E)$ is a graph $G^c = (V, E^c)$, where $E^c = \{ (v_1, v_2): v_1, v_2 \in V \text{ and } (v_1, v_2) \notin E \}$.

The reduction $f(G, k)$ from CLIQUE to IND-SET simply computes the complement of the graph and outputs $(G^c, k)$.

$f$ satisfies:

- $U$ is a clique in $G \iff U$ is a independent set in $G^c$.
- computable in polynomial time!

♣

**Remark 65**

Same reduction shows that IND-SET $\leq_P$ CLIQUE
Reductions to $\mathcal{P}$

**Theorem 66**

If $A \leq_P B$ and $B \in \mathcal{P}$ then $A \in \mathcal{P}$.

**Proof:**

- Let $f$ the reduction from $A$ to $B$, computed by TM $M_f$.
  On input $x$, the TM $M_f$ makes at most $c_f \cdot |x|^{a_f}$ steps.
- Let $M_B$ be the poly-time decider for $B$.
  On input $y$, the TM $M_B$ makes at most $c_B \cdot |y|^{a_B}$ steps.

**Algorithm 67 (Decider $M_A$ for $A$)**

On input $x$, return $M_B(f(x))$

- $M_A$ decides $A$
- Since $|f(x)| \leq c_f |x|^{a_f}$, running time of $M_B(x)$, is at most $c_B \cdot (c_f \cdot |x|^{a_f})^{a_B} = (c_B \cdot c_f^{a_B}) \cdot |x|^{a_f \cdot a_B} \in \text{poly}(|x|)$
  Hence, $A \in \mathcal{P}$
What $A \leq_p B$ tells us about $B$?

**Question 68**

Assume that $\{0^n1^n : n \geq 0\} \leq_p L$. Does it yield that $L \in \mathcal{P}$?

**Answer:** No. (Reduction in the wrong direction!)

Let $L = H_{TM, \varepsilon}$ and define $f(x) = \begin{cases} M_{stop}, & x \in \{0^n1^n : n \in \mathbb{N}\} \\ M_{no-stop}, & \text{otherwise.} \end{cases}$

$A \leq_p B$ does tell us that $B$ is “at least as hard” as $A$. 
NP completeness, formal definition

**Definition 69 (NP-complete)**

A language $B$ is NP-complete, if

- $B \in \text{NP}$, and
- Every $A \in \text{NP}$ is poly-time reducible to $B$ (i.e., $A \leq_P B$)

Let $\text{NPC}$ denote the class of all NP-complete languages.

Compare to

**Definition 70 (RE-complete)**

A language $B$ is RE-complete, if

- $B \in \text{RE}$, and
- Every $A \in \text{RE}$ is mapping reducible to $B$. 
Why NP completeness?

**Theorem 71**

If \( B \in \text{NPC} \) and \( B \in \text{P} \), then \( \text{P} = \text{NP} \).

**Proof:** Immediately follows by Thm 66. ♣

To show \( \text{P} = \text{NP} \) (and make an instant fortune, see [www.claymath.org/millennium/P_vs_NP/](http://www.claymath.org/millennium/P_vs_NP/)), suffices to find a polynomial-time algorithm for any NP-complete problem.

**Question 72**

Is \( \text{NPC} \) empty?
\( \text{Is not empty} \)

\[
A_{\text{NP}} = \{ \langle M, x, 1^n \rangle : M \text{ is a TM } \land \exists c \in \Sigma^* \text{ s.t. } M(x, c) \text{ accepts within } n \text{ steps} \}.
\]

**Theorem 73**

\( A_{\text{NP}} \in \mathcal{NPC} \)

**Proof:**

- Clearly \( A_{\text{NP}} \in \mathcal{NP} \).
- Let \( L \in \mathcal{NP} \), let \( V \) be a verifier for \( L \) and let \( p \in \text{poly} \) be a bound on the running time of \( V \) (i.e., \( V(x, \cdot) \) halts within \( p(|x|) \) steps, for every \( x \in \Sigma^* \)).
- Define \( f(x) = \langle V, x, 1^p(|x|) \rangle \).
- \( f \) is poly-time computable
- \( x \in L \iff f(x) \in A_{\text{NP}}. \)
Finding additional NP-complete languages

**Theorem 74**

Assume that

1. \( B \in \mathcal{NP} \)

2. \( A \in \mathcal{NPC} \) and \( A \leq_P B \)

then \( B \in \mathcal{NPC} \).

**Proof:** Home exercise …

We would like to find \( L \in \mathcal{NPC} \) that is “natural" and “easy" to reduce to.